

A Dirichlet Process Functional Approach to Heteroscedastic-Consistent Covariance Estimation

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Abstract

The mixture of Dirichlet process (MDP) defines a flexible prior distribution on the space of probability measures. This study shows that ordinary least-squares (OLS) estimator, as a functional of the MDP posterior distribution, has posterior mean given by weighted least-squares (WLS), and has posterior covariance matrix given by the (weighted) heteroscedastic-consistent sandwich estimator. This is according to a pairs bootstrap distribution approximation of the posterior, using a Pólya urn scheme. Also, when the MDP prior baseline distribution is specified as a product of independent probability measures, this WLS solution provides a new type of generalized ridge regression estimator. Such an estimator can handle multicollinear or singular design matrices even when the number of covariates exceeds the sample size, and can shrink the coefficient estimates of irrelevant covariates towards zero, which makes it useful for nonlinear regressions via basis expansions. Also, this MDP/OLS functional methodology can be extended to methods for analyzing the sensitivity of the heteroscedasticity-consistent causal effect size over a range of hidden biases due to missing covariates omitted from the regression, and more generally extended to a Vibration of Effects analysis. The methodology is illustrated through the analysis of simulated and real data sets. Overall, this study establishes new connections between Dirichlet process functional inference, the bootstrap, consistent sandwich covariance estimation, ridge shrinkage regression, WLS, and sensitivity analysis, to provide regression methodology useful for inferences of the mean dependent response.

Key words. Bayesian Nonparametric, Bootstrap, Regression, Sandwich Estimator, Causal Inference, Sensitivity Analysis.

1 Introduction

When the linear regression model is misspecified due to the presence of heteroscedasticity, the sampling covariance matrix of the ordinary least-squares (OLS) estimator of the regression coefficients becomes inconsistent. White's (1980) sandwich covariance matrix estimator is consistent even under heteroscedasticity, and does not require any modeling specification for the form of the heteroscedasticity (see also Eicker, 1963, 1967; Huber, 1967). Hence, the sandwich estimator is often referred to as heteroscedastic-consistent or -robust.

White's article has profoundly impacted applied statistics and econometrics. By June 2006, it was most cited by others in the peer-reviewed economics literature since 1970 (Kim et al., 2006), and cited over 21,700 times according to a May 2016 internet search. However, from a frequentist perspective, the sandwich estimator can exhibit downward bias for small sample size (n) data sets containing observations with high leverage on the OLS estimand (Chesher & Jewitt, 1987). This has led researchers to propose various leverage-adjusted sandwich estimators (see MacKinnon, 2013).

Recent studies have proposed Bayesian linear modeling methods that make use of the sandwich estimator. Müller (2013) showed that regression coefficient inference has lower asymptotic frequentist risk when using an artificial multivariate normal posterior distribution centered on the maximum likelihood estimate with sandwich covariance matrix, compared to the posterior distribution under the homoscedasticity assumption. Hoff and Wakefield (2013) and Startz (2015) extended this approach by incorporating informative prior distributions, and showed that the heteroscedastic-robust posterior can exhibit more uncertainty than the posterior under the homoscedasticity assumption. Norets (2015) proposed flexible Bayesian nonparametric (e.g., Gaussian process) models for the regression error terms, with the motivation that fully Bayesian nonparametric, Dependent Dirichlet process (DDP) infinite-mixture regression models (e.g., DeIorio et al. 2004; Dunson & Park, 2008) require a lot of data for reliable estimation results, and require prior specification that is non-trivial in practice. Generally speaking, each of these prior-informed Bayesian regression models requires use of an Markov chain Monte Carlo (MCMC) sampling algorithm for estimating the posterior distribution of the regression coefficients. MCMC can be computationally-intensive for a large data set. Further, it may be argued that if in a practical setting the primary aim is to perform heteroscedastic-consistent inferences of linear regression coefficients, then there is no point in using intensive Monte Carlo estimation methods because OLS and sandwich estimators can already be numerically evaluated (MacKinnon, 2013).

Lancaster (2003) showed that the OLS estimator, as a functional of the classical bootstrap (CB) (Efron, 1979) distribution, or of the Bayesian Bootstrap (BB) (Rubin, 1981) distribution, has covariance matrix that is order n^{-1} equivalent to the sandwich estimator. Here, we refer to the *pairs* bootstrap, where each of the n observations consists of the dependent variable observation paired with its corresponding observations on p covariates. In the CB, the n observations are assigned (single-trial) multinomial (re)sampling probabilities (weights) of $1/n$ (resp.). In the BB, these n sampling probabilities have a Dirichlet posterior distribution with concentration parameters 1 (resp.), under an improper non-informative prior. The pairs BB was studied by Chamberlain and Imbens (2003), Szpiro et al. (2010), and Taddy et al. (2015).

Poirier (2011) proposed a pairs BB approach that employs an informative Dirichlet prior distribution that can be chosen to assign positive support to data observations, and to imaginary observations. He showed that the prior-informed pairs BB distribution of the OLS estimator has posterior mean given by weighted least squares (WLS), and posterior covariance matrix given by a weighted sandwich estimator, according to a Taylor series approximation. He also showed that nearly all of the frequentist-based leverage-adjusted sandwich estimators (mentioned earlier) can be characterized as assuming a particular Dirichlet prior distribution that does not support imaginary observations, and can give rise to a posterior distribution that places too much support to extreme sampling probability weights. However, he observed that it is not necessarily easy in practice to elicit an informative Dirichlet prior that supports imaginary observations, and then concluded that more attractive informative prior specifications await further research.

The Dirichlet process (DP) defines a flexible prior distribution on the space of random probability measures (r.p.m.s) (distribution functions), and is parameterized by a baseline distribution and a precision parameter (α) which respectively control the mean and variance of the r.p.m. (Ferguson, 1973). The DP prior has the conjugacy property, in the sense that a data update of this prior leads to a posterior distribution for the r.p.m. that is also a DP; and the DP is the only process that has this conjugacy property in the class of homogeneous normalized random measures with independent increments (James, et al., 2006). Also, the BB's Dirichlet posterior distribution is the DP posterior distribution under a non-informative DP prior with limiting zero precision parameter.

Cifarelli and Regazzini (1979) initiated a line of research that deals with the problem of determining the expression for the distribution of functionals of the Dirichlet process, with any prescribed

error of approximation (for reviews, see Regazzini et al., 2002; Lijoi & Prunster, 2009). This research has primarily focused the mean and other linear functionals of the DP.

This article studies approximations of the distribution of the OLS estimator as a functional of the mixture of Dirichlet process (MDP) posterior distribution. The MDP prior is a DP prior, with a hyperprior distribution (at least) on the precision parameter (Antoniak, 1974). Conditionally on the precision and baseline parameters, the DP posterior distribution can be well-approximated by a bootstrap distribution that is defined by the Pólya urn scheme characterization of the DP posterior (Blackwell & MacQueen, 1973). Specifically, if each resampled data set of the bootstrap procedure has sample size $n + \alpha + 1$, then the DP posterior and bootstrap distributions have identical means and variances (Hjort, 1985). This equality can be directly verified analytically, thanks to the DP conjugacy property that allows for explicit expressions of the posterior DP mean and variance. As a consequence, for any well-behaved functional, including the OLS estimator, the bootstrap distribution of the functional (via the Pólya urn scheme) well-approximates the DP posterior distribution of the functional (Hjort, 1985). By extension, this is true for the MDP posterior distribution of the functional, after marginalizing out the posterior distribution of the precision parameter. In this study we focus on the DP (MDP) because it is the only Gibbs-type prior that enables posterior consistency for either continuous or discrete r.p.m.s (De Blasi et al., 2015), while regression applications often involve the use of continuous variables.

Ferguson (1973, p. 209) introduced the DP prior with the motivation that the DP posterior distribution "should be manageable analytically," and that the "support of the prior distribution should be large-with respect to some suitable topology on the space of probability distributions on the sample space." He then provided explicit analytical solutions to a list of nonparametric statistical problems based on the DP posterior, including the estimation of a distribution function, median, quantiles, variance, covariance, and the probability that one variable exceeds another.

The current article adds to his list by showing that the OLS estimator, as a functional of the MDP posterior distribution, has posterior mean given by WLS, and posterior covariance matrix given by a weighted heteroscedastic-consistent sandwich estimator. This is according to a Taylor series approximation of the pairs bootstrap (MDP posterior predictive) distribution, using the multivariate delta method. Under a non-informative DP prior, this sandwich estimator closely approximates White's (1980) original (unweighted) sandwich estimator. Also, it is shown that if the MDP prior baseline distribution is specified as a product of independent probability measures, then this WLS solution is the Bayesian generalized ridge regression estimator (Hoerl & Kennard, 1970). It is known that such an estimator can handle multicollinear or singular covariate design matrices, even when the number of covariates exceeds the sample size (i.e., $p > n$), while shrinking the coefficient estimates of irrelevant covariates towards zero. These features of ridge regression are useful for fitting nonlinear regressions via basis expansions, and further ridge regression is tough to beat in terms of predictive power (Griffin & Brown, 2013). Clearly, these posterior quantities (WLS and sandwich estimators) are analytically manageable and permit fast computations even for large data sets. The current study is the first to draw connections between the DP and Bayesian ridge regression, and to provide heteroscedastic-consistent covariance estimation for ridge regression.

The following sections elaborate on the main findings of this article. Section 2 describes the specific MDP model that is employed, and presents the model's key conditional and marginal posterior distributions. This includes the posterior of the precision parameter as give by Nandram and Choi (2004). Section 3 briefly reviews the key properties and assumptions of the OLS and sandwich estimators. It then establishes connections between the OLS estimator, ridge regression, and the posterior moments of the MDP model, using imputation methods for imaginary data that has the chosen baseline distribution for the MDP. Section 4 provides details about how the OLS functional of the MDP posterior is approximated by the bootstrap distribution of this functional.

In practice, if the OLS (WLS) estimate of the regression coefficients is inconsistent or biased, then the heteroscedastic-consistency of the sandwich covariance estimator can become meaningless (Freedman, 2006). This inconsistency results from correlation between covariates and regression errors, implying a violation of the exogeneity assumption of regression (Greene, 2012) and the presence of hidden bias due to missing covariates ("confounders") omitted from the regression equation (Rosenbaum, 2002). Section 5 describes how the MDP/OLS functional methodology can easily incorporate methods of sensitivity analysis (van der Weele & Arah, 2011), which aim to evaluate how much the causal effect size, of a covariate of interest, varies over a hypothesized range of hidden biases. In the current setting, the effect size is defined by the ratio of the slope coefficient estimate of the covariate, over its heteroscedastic-consistent posterior standard deviation.

Generally speaking, the MDP/OLS methodology can incorporate a vibration of effects (VoE) analysis (Ioannidis, 2008) in order to assess how much the effect size differs (or vibrates) over different ways that the analysis can be performed, for e.g., with respect to different: variables that are included and excluded in the analysis (statistical adjustments); models used; definitions of outcomes and predictors; and inclusion and exclusion criteria for the study population. VoE analysis addresses the fact that an effect size estimator can display noticeable variance (vibration) over different ways that the data analysis is done. This variance can lead to bias if only a few chosen analyses are reported, especially if the investigators have a preference for a particular result or are influenced by optimism bias (Ioannidis et al. 2014, p.168). A recent study proposed a VoE analysis method for regression settings (Patel, et al. 2015), which entails studying the variance of the effect size over all different subsets of $K - 1$ other (adjustment) covariates that may be included in the regression. But as noted, this full enumeration approach is infeasible for sufficiently large K . In this study consider a VoE analysis approach that employs the Least Angle Regression (LARS) algorithm (Efron, et al. 2004). LARS provides a fast and directed selection of covariates, yielding a path of $K + 1$ regression solutions that include $k = 0, \dots, K$ covariates in the regression equation (resp.), at the computational cost of a single OLS fit.

Section 6 describes a simulation study that evaluates the MDP/OLS functional methodology in terms of coverage rates of 95% posterior intervals of linear regression coefficients. These rates are studied over a range of conditions of sample size, covariate distribution, degree of heteroscedasticity, and choice of prior distribution for the MDP precision parameter. Section 7 illustrates the functional methodology on two real data sets. Section 8 concludes with some suggestions for future research, including extensions of the methodology to other Bayesian nonparametric priors.

2 Mixture of Dirichlet Process Model

Let $\mathbf{Z}_n = (\mathbf{z}_i^\top = (\mathbf{x}_i^\top, y_i))_{i=1}^n = (\mathbf{X}, \mathbf{y}) = (\mathbf{X}_n, \mathbf{y}_n)$ denote a data set (matrix) of n observations of the variable $\mathbf{Z} = (\mathbf{X}, Y)$, including a dependent variable Y and K covariates $\mathbf{x} = (x_1, \dots, x_k, \dots, x_K)^\top$. The data set has $c_n \leq n$ distinct values (clusters) $\mathbf{Z}_{c_n}^* = (\mathbf{X}_{c_n}^*, \mathbf{y}_{c_n}^*) = (\mathbf{z}_c^{*\top} = (\mathbf{x}_c^{*\top}, y_c^*))_{c=1}^{c_n \leq n}$, with frequency counts $\mathbf{n}_{c_n} = (n_1, \dots, n_{c_n})^\top$ (resp.), and $\sum_{c=1}^{c_n} n_c = n$. Such a data set is assumed to consist of n exchangeable samples from an unknown distribution function F , having space $\mathcal{F}_{\mathbf{Z}} = \{F\}$, the set of all probability measures on $\mathcal{Z} = \{\mathbf{Z}\} \subset \mathbb{R}^{K+1}$, according to the MDP model:

$$\mathbf{z}_i | F \sim F, \quad i = 1, \dots, n, \quad (1a)$$

$$F | \alpha \sim \mathcal{DP}(\alpha, F_0), \quad (1b)$$

$$F_0(\mathbf{z}) = N_{K+1}(\mathbf{x}_i^\top, y_i | \mathbf{m}_{\mathbf{z}}, \mathbf{V}_{\mathbf{z}}) \quad (1c)$$

$$\alpha \sim \pi(\alpha), \quad (\alpha > 0). \quad (1d)$$

$\mathcal{DP}(\alpha, F_0)$ denotes the Dirichlet process (DP) prior distribution on $\mathcal{F}_{\mathcal{Z}}$, with precision parameter α , and baseline distribution F_0 , specified as a $(K+1)$ -variate normal distribution with mean vector parameter $\mathbf{m}_{\mathbf{z}} = (m_k)_{k=1}^{K+1} = (\mathbf{m}_{\mathbf{x}}, m_Y)^\top$ and covariance matrix parameter $\mathbf{V}_{\mathbf{z}}$.

In the current study, we focus on the *ridge baseline prior*, defined by:

$$F_0(\mathbf{z}) = \mathcal{N}(x_1 | 0, 0) \prod_{k=2}^K \mathcal{N}(x_k | 0, v_{\mathbf{x}k}) \mathcal{N}(y | 0, 0), \quad (2)$$

implying $\mathbf{m}_{\mathbf{z}} = (\mathbf{m}_{\mathbf{x}}, m_Y)^\top = \mathbf{0}_{K+1}^\top$, $\mathbf{V}_{\mathbf{z}} = \text{diag}(0, v_{\mathbf{x}2}, \dots, v_{\mathbf{x}K}, 0)$, with $\mathbf{0}_K$ a column vector of K zeros. The *unit ridge baseline prior* further assumes $\mathbf{V}_{\mathbf{z}} = \text{diag}(0, \mathbf{1}_{K-1}^\top, 0)$, with $\mathbf{1}_K$ a column vector of K ones. This study finds that each of these baseline distributions, along with α , has connections with ridge regression, for reasons given in the next section.

The precision parameter α in (1d) of the MDP model (1) represents a (prior) sample size for the number of imaginary observations of \mathbf{z} , and is assigned a prior distribution with p.d.f. $\pi(\alpha)$. In this study we consider the uniform prior p.d.f. $\text{un}(\alpha | 0, \xi) = \mathbf{1}(0 < \alpha < \xi)/\xi$, where $\mathbf{1}(\cdot)$ is the indicator function; as well as a ξ -truncated version of a Cauchy-type shrinkage prior p.d.f. $\pi(\alpha) = \mathbf{1}(0 < \alpha < \xi)/(\alpha + 1)^2$, $\alpha > 0$ (Nandram & Yin, 2016).

The conditional DP prior distribution $F | \alpha \sim \mathcal{DP}(\alpha, F_0)$ has a Dirichlet (Di) distribution:

$$F(B_1), \dots, F(B_k) | \alpha \sim \text{Di}_k(\alpha F_0(B_1), \dots, \alpha F_0(B_k)), \quad (3)$$

for all $k \geq 1$ partitions B_1, \dots, B_k of \mathcal{Z} , with prior mean $\mathbb{E}[F(B) | \alpha] = F_0(B)$ and variance $\mathbb{V}[F(B) | \alpha] = F_0(B)\{1 - F_0(B)\}/(\alpha + 1)$ for $\forall B \in \mathcal{B}(\mathcal{Z})$ (Ferguson, 1973). The probability (likelihood) distribution for the number of clusters is given by (Antoniak, 1974):

$$P(C_n = k | \alpha) = \frac{s_n(k) \alpha^k \Gamma(\alpha)}{\Gamma(\alpha + n)}, \quad (\alpha > 0), \quad (4)$$

where the $s_n(k)$ are the signless Stirling numbers of the first kind (Abramowitz & Stegun, 1965).

The conditional posterior distribution of F is also a DP, with $F | \mathbf{Z}_n, \alpha \sim \mathcal{DP}(\alpha, F_0)$, and has Dirichlet distribution:

$$F(B_1), \dots, F(B_k) | \mathbf{Z}_n, \alpha \sim \text{Di}_k(\alpha F_0(B_1) + n \hat{F}_n(B_1), \dots, \alpha F_0(B_k) + n \hat{F}_n(B_k)), \quad (5)$$

for all $k \geq 1$ partitions B_1, \dots, B_k of \mathcal{Z} , with baseline distribution F_0 (1c); $\hat{F}_n(\cdot) = \sum_{c=1}^{c_n} \frac{n_c}{n} \delta_{\mathbf{z}_c^*}(\cdot)$ is the empirical distribution function (e.d.f.) of the data, \mathbf{Z}_n ; and $\delta_{\mathbf{z}}(\cdot)$ is the degenerate probability measure $\delta_{\mathbf{z}}(\mathbf{z}) = 1$, with $\delta_{\mathbf{z}}(B) = 1$ if $\mathbf{z} \in B$ (Ferguson, 1973). This posterior distribution has conditional expectation and variance (resp.):

$$\mathbb{E}[F(B) | \mathbf{Z}_n, \alpha] = \Pr(\mathbf{z}_{n+1} \in B | \mathbf{Z}_n, \alpha) \quad (6a)$$

$$= \frac{n}{\alpha + n} \hat{F}_n(B) + \frac{\alpha}{\alpha + n} F_0(B) := \bar{F}_\alpha(B), \quad (6b)$$

$$= \sum_{c=1}^{c_n} \frac{n_c}{\alpha + n} \delta_{\mathbf{z}_c^*}(B) + \frac{\alpha}{\alpha + n} F_0(B); \quad (6c)$$

$$\mathbb{V}[F(B) | \mathbf{Z}_n, \alpha] = \frac{\bar{F}_\alpha(B)\{1 - \bar{F}_\alpha(B)\}}{\alpha + n + 1}, \quad \forall B \in \mathcal{B}(\mathcal{Z}). \quad (6d)$$

The posterior expectation (6a)-(6c) gives the posterior predictive probability of a new observation, $\mathbf{z}_{n+1} = (\mathbf{x}_{n+1}^\top, y_{n+1})^\top \in \mathcal{Z}$, according to the Pólya urn scheme (Blackwell & MacQueen, 1973). This scheme states that with probability $\frac{n_c}{\alpha + n}$, a new observation $\mathbf{z}_{n+1} = (\mathbf{x}_{n+1}, y_{n+1})$ takes on value $\mathbf{z}_c^* = (\mathbf{x}_c^*, y_c^*)$ of an existing cluster c , for $c = 1, \dots, c_n$; and otherwise with probability $\frac{\alpha}{\alpha + n}$, the new observation $\mathbf{z}_{n+1} = (\mathbf{x}_{n+1}, y_{n+1})$ is a sample $\mathbf{z}_{c_n+1}^*$ from the baseline distribution F_0 ,

(1c). Equations (6b)-(6c) in particular show that the conditional posterior predictive distribution function $\Pr(\mathbf{z}_{n+1} \in B | \mathbf{Z}_n, \alpha)$ is a linear combination of two data sets, namely, the empirical data set \mathbf{Z}_n with empirical distribution function $\hat{F}_n(B)$, and an imaginary data set having distribution function F_0 , with sample sizes (weights) n and α (resp.). We will revisit this point in the next section.

Finally, the posterior distribution $\Pi(\alpha | \mathbf{Z}_n)$ has p.d.f.:

$$\pi(\alpha | c_n) \propto \pi(\alpha) \alpha^{c_n} \Gamma(\alpha) / \Gamma(\alpha + n), \quad (7)$$

up to a normalization constant (Nandram & Choi, 2004, p. 828).

3 Review of OLS properties, and Connections with MDP model

Now we briefly review of the key properties of the OLS estimator for the linear model (for more details, see Greene, 2012). Then we present connections between the OLS estimator, ridge regression, and posterior inference with the MDP model. This will set up the discussion in the next section on the MDP-based bootstrap procedure, and the associated (WLS) posterior mean and heteroscedastic-consistent posterior covariance matrix estimator.

The linear regression equation, defined by $y_i = \beta_1 x_{i1} + \dots + \beta_K x_{iK} + \varepsilon_i = \mathbf{x}_i^\top \boldsymbol{\beta}$ for observations indexed by $i = 1, \dots, n$, has regression errors $(\varepsilon_1, \dots, \varepsilon_i, \dots, \varepsilon_n)$ assuming corresponding variances $\Phi = \text{diag}(\sigma_1^2, \dots, \sigma_n^2)$, and assuming exogeneity, i.e., $\mathbb{E}[\varepsilon_j | \mathbf{x}_i] = 0$, for $i, j = 1, \dots, n$. The OLS estimator of the coefficients $\boldsymbol{\beta}$ is given by:

$$\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}_n = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y} = (\mathbf{X}_{c_n}^{*\top} \text{diag}(\mathbf{n}_{c_n}) \mathbf{X}_{c_n}^*)^{-1} \mathbf{X}_{c_n}^{*\top} \text{diag}(\mathbf{n}_{c_n}) \mathbf{y}_{c_n}^*,$$

and has sampling covariance matrix:

$$\mathbb{V}(\hat{\boldsymbol{\beta}}) = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \Phi \mathbf{X} (\mathbf{X}^\top \mathbf{X})^{-1}. \quad (8)$$

If homoscedasticity holds (i.e., $\sigma^2 = \sigma_1^2 = \dots = \sigma_n^2$), then $\Phi = \sigma^2 \mathbf{I}_n$ and $\mathbb{V}(\hat{\boldsymbol{\beta}}) = \sigma^2 (\mathbf{X}^\top \mathbf{X})^{-1}$, and $\hat{\sigma}^2 (\mathbf{X}^\top \mathbf{X})^{-1}$ provides a consistent estimator of $\mathbb{V}(\hat{\boldsymbol{\beta}})$ with $\hat{\sigma}^2 = (\frac{1}{n-K})(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})^\top (\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})$, but is inconsistent otherwise. The (finite-sample) heteroscedastic-consistent (sandwich) estimator of $\mathbb{V}(\hat{\boldsymbol{\beta}})$ is given by (White, 1980):

$$\text{HC0} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \text{diag}(\hat{u}_1^2, \dots, \hat{u}_n^2) \mathbf{X} (\mathbf{X}^\top \mathbf{X})^{-1}, \quad (9)$$

where $\hat{u}_i = \hat{\varepsilon}_i = y_i - \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}$ for $i = 1, \dots, n$, which reduces to $\text{HC0} = \hat{\sigma}^2 (\mathbf{X}^\top \mathbf{X})^{-1}$ under homoscedasticity. This consistency does not rely on exogeneity. Asymptotically ($n \rightarrow \infty$), $n^{1/2}(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}) \xrightarrow{L} N_K(\mathbf{0}, n\mathbb{V}(\hat{\boldsymbol{\beta}}))$ in law under mild conditions, and $n\text{HC0}$ consistently estimates $n\mathbb{V}(\boldsymbol{\beta})$.

Recall from (6a)-(6c) that the conditional posterior expectation $\mathbb{E}[F(\cdot) | \mathbf{Z}_n, \alpha]$ under the MDP model is a linear combination of two distribution functions, \hat{F}_n and F_0 , corresponding to two data sets (resp.) of total sample size $\alpha + n$. The empirical c.d.f., \hat{F}_n , which describes the data set $(\mathbf{X}_n, \mathbf{y}_n)$, has sample mean vector $\hat{\mathbf{m}}_{\mathbf{z}} = (\hat{\mathbf{m}}_{\mathbf{x}}^\top, \hat{m}_Y)^\top$, and $(K+1) \times (K+1)$ covariance matrix $\hat{\mathbf{V}}_{\mathbf{z}}$, including the $K \times K$ covariance matrix $\hat{\mathbf{V}}_{\mathbf{x}}$ of \mathbf{X} and the $K \times 1$ vector $\hat{\mathbf{V}}_{\mathbf{x}Y}$ of covariances between the columns of \mathbf{X} and \mathbf{y} (resp.). The MDP baseline distribution, F_0 (in (1c)), which describes the distribution of imaginary data set of S observations, given by $\bar{\mathbf{Z}}_S = (\bar{\mathbf{X}}_S, \bar{\mathbf{y}}_S) = ((\bar{x}_{sk})_{S \times K}, (\bar{y}_{sk})_{S \times 1})$, has baseline mean $\mathbf{m}_{\mathbf{z}} = (\mathbf{m}_{\mathbf{x}}^\top, m_Y)^\top$ and $(K+1) \times (K+1)$ covariance matrix $\mathbf{V}_{\mathbf{z}}$, including the $K \times K$ covariance matrix $\mathbf{V}_{\mathbf{x}}$ of \mathbf{X} and the $K \times 1$ vector $\mathbf{V}_{\mathbf{x}Y}$ of covariances of each column of \mathbf{X} with Y (resp.).

Let $(\mathbb{X}, \mathbb{Y}) = \left(\begin{smallmatrix} \mathbf{X}_{c_n}^* \\ \bar{\mathbf{X}}_S \end{smallmatrix}, \begin{smallmatrix} \mathbf{y}_{c_n}^* \\ \bar{\mathbf{y}}_S \end{smallmatrix} \right)$. If F_0 is a general baseline distribution in the MDP model, perhaps not a ridge baseline (2), then the OLS estimator $\hat{\beta}$ from the data (\mathbb{X}, \mathbb{Y}) satisfies the equalities:

$$\hat{\beta} = \left[\left(\begin{smallmatrix} \mathbf{X} \\ \bar{\mathbf{X}}_S \end{smallmatrix} \right)^\top \text{diag}(\mathbf{1}_n^\top, (\frac{\alpha}{S})\mathbf{I}_S) \left(\begin{smallmatrix} \mathbf{X} \\ \bar{\mathbf{X}}_S \end{smallmatrix} \right) \right]^{-1} \left(\begin{smallmatrix} \mathbf{X} \\ \bar{\mathbf{X}}_S \end{smallmatrix} \right)^\top \text{diag}(\mathbf{1}_n^\top, (\frac{\alpha}{S})\mathbf{I}_S) \left(\begin{smallmatrix} \mathbf{y} \\ \bar{\mathbf{y}}_S \end{smallmatrix} \right) \quad (10a)$$

$$= \left[\left(\begin{smallmatrix} \mathbf{X}_{c_n}^* \\ \bar{\mathbf{X}}_S \end{smallmatrix} \right)^\top \text{diag}(\mathbf{n}_{c_n}^\top, (\frac{\alpha}{S})\mathbf{I}_S) \left(\begin{smallmatrix} \mathbf{X}_{c_n}^* \\ \bar{\mathbf{X}}_S \end{smallmatrix} \right) \right]^{-1} \left(\begin{smallmatrix} \mathbf{X}_{c_n}^* \\ \bar{\mathbf{X}}_S \end{smallmatrix} \right)^\top \text{diag}(\mathbf{n}_{c_n}^\top, (\frac{\alpha}{S})\mathbf{I}_S) \left(\begin{smallmatrix} \mathbf{y}_{c_n}^* \\ \bar{\mathbf{y}}_S \end{smallmatrix} \right) \quad (10b)$$

$$= (\mathbb{X}^\top \text{diag}(\mathbf{n}^\top, (\frac{\alpha}{S})\mathbf{I}_S) \mathbb{X})^{-1} \mathbb{X}^\top \text{diag}(\mathbf{n}^\top, (\frac{\alpha}{S})\mathbf{I}_S) \mathbb{Y} \quad (10c)$$

$$= \left(\mathbb{X}^\top \left[\frac{1}{\alpha+n} \text{diag}(\mathbf{n}^\top, (\frac{\alpha}{S})\mathbf{I}_S) \right] \mathbb{X} \right)^{-1} \mathbb{X}^\top \left[\frac{1}{\alpha+n} \text{diag}(\mathbf{n}^\top, (\frac{\alpha}{S})\mathbf{I}_S) \right] \mathbb{Y} \quad (10d)$$

$$= \left(n(\widehat{\mathbf{V}}_{\mathbf{x}} + \widehat{\mathbf{m}}_{\mathbf{x}}\widehat{\mathbf{m}}_{\mathbf{x}}^\top) + \alpha(\mathbf{V}_{\mathbf{x}} + \mathbf{m}_{\mathbf{x}}\mathbf{m}_{\mathbf{x}}^\top) \right)^{-1} \left(n(\widehat{\mathbf{V}}_{\mathbf{xY}} + \widehat{m}_Y\widehat{\mathbf{m}}_{\mathbf{x}}) + \alpha(\mathbf{V}_{\mathbf{xY}} + m_Y\mathbf{m}_{\mathbf{x}}) \right) \quad (10e)$$

where in (10a)-(10d), the diagonal elements of $(\alpha/S)\mathbf{I}_S$ sum to α , the number of prior imaginary observations. Also, (10e) still yields the OLS estimator $\hat{\beta}$ after replacing n with $\frac{n}{\alpha+n}$, and replacing α with $\frac{\alpha}{\alpha+n}$. If α is a positive integer with $S = \alpha$, then $(\alpha/S)\mathbf{I}_S = (\alpha/\alpha)\mathbf{I}_\alpha = \mathbf{I}_\alpha$, the diagonal elements of \mathbf{I}_α sum to α , and the OLS estimator (10) has the familiar form, given by $\hat{\beta} = (\mathbf{X}_{n+\alpha}^\top \mathbf{X}_{n+\alpha})^{-1} \mathbf{X}_{n+\alpha}^\top \mathbf{y}_{n+\alpha}$, where $(\mathbf{X}_{n+\alpha}, \mathbf{y}_{n+\alpha}) = \left(\begin{smallmatrix} \mathbf{X}_n & \mathbf{y}_n \\ \bar{\mathbf{X}}_\alpha & \bar{\mathbf{y}}_\alpha \end{smallmatrix} \right)$.

Given any precision parameter $\alpha > 0$, where possibly $S \neq \alpha$, an extension of the fractional imputation procedure (e.g., Kim & Kim, 2012) can be used to simulate the imaginary data set $(\bar{\mathbf{X}}, \bar{\mathbf{y}})$. This would involve taking a large number (S) of Monte Carlo sample draws $\mathbf{z}_{c_n+1,s} = (\mathbf{x}_{c_n+1,s}, y_{c_n+1,s}) \sim F_0$, for $s = 1, \dots, S$, to provide the Monte Carlo estimator $\frac{1}{S} \sum_{s=1}^S \mathbf{z}_{c_n+1,s} \mathbf{z}_{c_n+1,s}^\top = (\mathbf{V}_{\mathbf{z}} + \widehat{\mathbf{m}}_{\mathbf{z}}\widehat{\mathbf{m}}_{\mathbf{z}}^\top)$, including $(\mathbf{V}_{\mathbf{x}} + \widehat{\mathbf{m}}_{\mathbf{x}}\widehat{\mathbf{m}}_{\mathbf{x}}^\top)$ and $(\mathbf{V}_{\mathbf{xY}} + \widehat{m}_Y\widehat{\mathbf{m}}_{\mathbf{x}})$, since $\text{plim}_{S \rightarrow \infty} \frac{1}{S} \sum_{s=1}^S \mathbf{z}_{c_n+1,s} \mathbf{z}_{c_n+1,s}^\top = (\mathbf{V}_{\mathbf{z}} + \mathbf{m}_{\mathbf{z}}\mathbf{m}_{\mathbf{z}}^\top)$ by the law of large numbers. Then, the OLS estimator (10) is obtained after plugging in the simulated data $(\mathbf{x}_{c_n+1,s}^\top)_{s=1}^S$ in place of $\bar{\mathbf{X}}_S$ and plugging in $(y_{c_n+1,s})_{s=1}^S$ in place of $\bar{\mathbf{y}}_S$ in (10b), using fractional weights $(\alpha/S)\mathbf{1}_S$; or by plugging in $(\mathbf{V}_{\mathbf{x}} + \widehat{\mathbf{m}}_{\mathbf{x}}\widehat{\mathbf{m}}_{\mathbf{x}}^\top)$ and $(\mathbf{V}_{\mathbf{xY}} + \widehat{m}_Y\widehat{\mathbf{m}}_{\mathbf{x}})$ in place of $(\mathbf{V}_{\mathbf{x}} + \mathbf{m}_{\mathbf{x}}\mathbf{m}_{\mathbf{x}}^\top)$ and $(\mathbf{V}_{\mathbf{xY}} + m_Y\mathbf{m}_{\mathbf{x}})$ (resp.) in (10e).

If F_0 is chosen as the ridge baseline prior (2), then the OLS estimator (10) is equal to:

$$\hat{\beta} = \left[\left(\begin{smallmatrix} \mathbf{X}_{c_n}^* \\ \bar{\mathbf{X}}_K \end{smallmatrix} \right)^\top \text{diag}(\mathbf{n}_{c_n}^\top, \alpha \mathbf{I}_K) \left(\begin{smallmatrix} \mathbf{X}_{c_n}^* \\ \bar{\mathbf{X}}_K \end{smallmatrix} \right) \right]^{-1} \left(\begin{smallmatrix} \mathbf{X}_{c_n}^* \\ \bar{\mathbf{X}}_K \end{smallmatrix} \right)^\top \text{diag}(\mathbf{n}_{c_n}^\top, \alpha \mathbf{I}_K) \left(\begin{smallmatrix} \mathbf{y}_{c_n}^* \\ \bar{\mathbf{y}}_K \end{smallmatrix} \right) \quad (11a)$$

$$= \left[\left(\begin{smallmatrix} (n_c^{1/2} \mathbf{x}_{c_k}^*)_{c_n \times K} \\ \alpha^{1/2} \text{diag}(0, v_{\mathbf{x}2}, \dots, v_K)^{1/2} \end{smallmatrix} \right)^\top \left(\begin{smallmatrix} (n_c^{1/2} \mathbf{x}_{c_k}^*)_{c_n \times K} \\ \alpha^{1/2} \text{diag}(0, v_{\mathbf{x}2}, \dots, v_K)^{1/2} \end{smallmatrix} \right) \right]^{-1} \left(\begin{smallmatrix} n_c^{1/2} \mathbf{y}_{c_n}^* \\ \mathbf{0}_K \end{smallmatrix} \right) \quad (11b)$$

$$= \left[\left(\begin{smallmatrix} \mathbf{X} \\ \alpha^{1/2} \text{diag}(0, v_{\mathbf{x}2}, \dots, v_K)^{1/2} \end{smallmatrix} \right)^\top \left(\begin{smallmatrix} \mathbf{X} \\ \alpha^{1/2} \text{diag}(0, v_{\mathbf{x}2}, \dots, v_K)^{1/2} \end{smallmatrix} \right) \right]^{-1} \left(\begin{smallmatrix} \mathbf{y} \\ \mathbf{0}_K \end{smallmatrix} \right), \quad (11c)$$

$$\bar{\mathbf{X}}_K = \text{diag}(\mathbf{V}_{\mathbf{x}} + \mathbf{m}_{\mathbf{x}}\mathbf{m}_{\mathbf{x}}^\top)^{1/2} = \text{diag}(0, v_{\mathbf{x}2}, \dots, v_K)^{1/2}; \quad (11d)$$

$$\bar{\mathbf{y}}_K = (\mathbf{V}_{\mathbf{xY}} + m_Y\mathbf{m}_{\mathbf{x}})^\top = (\mathbf{0}_K + \mathbf{0}_K)^\top = \mathbf{0}_K, \quad (11e)$$

with imaginary data $(\bar{\mathbf{X}}_K, \bar{\mathbf{y}}_K)$, obtained by deterministic single imputation, without any simulation or Monte Carlo error. Then the OLS estimator (10) corresponds to the generalized ridge regression estimator (Hoerl & Kennard, 1970) with shrinkage parameters $\alpha \cdot (0, v_2, \dots, v_K)$. Further, if the unit ridge baseline prior is chosen, with $\mathbf{V}_{\mathbf{x}} = (0, \mathbf{1}_{K-1}^\top)$, then $\alpha^{1/2} \bar{\mathbf{X}}_K = \alpha^{1/2} \cdot \{(\mathbf{V}_{\mathbf{x}} + \mathbf{m}_{\mathbf{x}}\mathbf{m}_{\mathbf{x}}^\top)\}^{1/2} = \alpha^{1/2} \mathbf{1}_K^\top$ in terms of (11a)-(11c), then the OLS estimator (10) coincides with the ridge regression estimator with coefficient shrinkage parameter α (Hoerl & Kennard, 1970), as Hastie et al. (2009,

p. 96) observed without any reference to the DP. Finally, under the ridge baseline prior, we may just set $(\mathbb{X}, \mathbb{Y}) := \left(\frac{\mathbf{X}_{c_n}^*}{\bar{\mathbf{X}}_K}, \frac{\mathbf{y}_{c_n}^*}{\bar{\mathbf{y}}_K} \right)$.

4 Bootstrap Approximation to the MDP Posterior Distribution

Generally speaking, using an MDP model, it is possible to employ a bootstrap procedure for the inference of a random posterior functional, say $\phi := \phi(F)$, having c.d.f. $G(\mathbf{t} | \mathbf{Z}_n) = \Pr(\phi(F) \leq \mathbf{t} | \mathbf{Z}_n)$, marginally over the posterior distribution of α . Conditionally on a posterior draw $\alpha \sim \pi(\alpha | c_n)$, this procedure approximates $G(\mathbf{t} | \mathbf{Z}_n, \alpha)$ by the c.d.f. $G^*(\mathbf{t} | \mathbf{Z}_n, \alpha) = \Pr(\phi(F^*) \leq \mathbf{t} | \mathbf{Z}_n, \alpha)$, estimated by $\hat{G}^*(\mathbf{t} | \mathbf{Z}_n, \alpha) = \frac{1}{B} \sum_{b=1}^B \mathbf{1}(\phi(F_b^*) \leq \mathbf{t})$ given a large number B of bootstrap samples $\{F_b^*\}_{b=1}^B$. Specifically, each random c.d.f. F_b^* is constructed by:

$$F_b^*(\mathbf{t}) = \sum_{c=1}^{c_n+1} \frac{n_{cb}^*}{n+\alpha+1} \mathbf{1}(\mathbf{z}_c^* \leq \mathbf{t}) + \frac{(\alpha - \text{floor}(\alpha))}{n+\alpha+1} \mathbf{1}(\mathbf{z}_{c_n+1+1(\text{ceil}(\alpha) > \alpha), b}^* \leq \mathbf{t}), \quad (12)$$

given $n + \text{ceil}(\alpha) + 1$ draws of \mathbf{z}_c^* from the posterior predictive distribution $\mathbb{E}[F(\cdot) | \mathbf{Z}_n, \alpha] = \sum_{c=1}^{c_n} \frac{n_c}{\alpha+n} \delta_{\mathbf{z}_c^*}(\cdot) + \frac{\alpha}{\alpha+n} F_0(\cdot)$ given by (6c), and given a draw $\mathbf{n}_{c_n+1,b}^* = (n_{cb}^*)_{(c_n+1) \times 1}$ from a multinomial distribution having $n + \text{ceil}(\alpha) + 1$ trials and event probabilities $(\frac{n_1}{\alpha+n}, \dots, \frac{n_c}{\alpha+n}, \dots, \frac{n_{c_n}}{\alpha+n}, \frac{\alpha}{\alpha+n})$ (Hjort, 1985). Here, $\text{floor}(\cdot)$ and $\text{ceil}(\cdot)$ refer to the floor and ceiling functions. The last term in (12) ensures that the effective number of multinomial trials is $n + \alpha + 1$, whether or not $\alpha > 0$ is a positive integer. If each bootstrap sample has (effective) size $n + \alpha + 1$, then the conditional posterior expectation and variance, $\mathbb{E}[F^*(B) | \mathbf{Z}_n, \alpha]$ and $\mathbb{V}[F^*(B) | \mathbf{Z}_n, \alpha]$ (for $\forall B \in \mathcal{B}(\mathcal{Z})$) equals to that of (6a)-(6c) and (6d) for F ; while F has twice the skewness of F^* but is small (Hjort, 1985). Then for well-behaved functionals $\phi := \phi(F)$, the posterior distributions of $\phi(F)$ and of $\phi(F^*)$ are nearly equal, i.e., $G(\cdot | \mathbf{Z}_n, \alpha) \doteq G^*(\cdot | \mathbf{Z}_n, \alpha)$, given α (Hjort, 1985), and $G(\cdot | \mathbf{Z}_n) \doteq G^*(\cdot | \mathbf{Z}_n)$ marginally over the posterior $\pi(\alpha | c_n)$. This justifies a MDP-based approach to the bootstrap.

Using the MDP-based bootstrap, and extending ideas from Section 3, we perform inference of the posterior mean and covariance matrix of the random functional $\phi(F^*) := \beta(F^*)$, chosen as the OLS estimator for linear regression. In this case, a bootstrap replication of the OLS estimator, $\hat{\beta}(F^*)$, is given by the following sampling scheme:

$$\hat{\beta}(F^*) = \hat{\beta}(\mathbf{n}^*) = (\mathbb{X}^\top \text{diag}(\mathbf{n}^*) \mathbb{X})^{-1} \mathbb{X}^\top \text{diag}(\mathbf{n}^*) \mathbb{Y}, \quad (13a)$$

$$\mathbf{n}^* = \frac{n+\alpha+1}{n+\text{ceil}(\alpha)+1} (n_{cb}^{**})_{(c_n+1) \times 1} \quad (13b)$$

$$(n_{cb}^{**})_{(c_n+1) \times 1} | \mathbf{Z}_n, \alpha \sim \text{Mu}_{c_n+S}(n + \text{ceil}(\alpha) + 1; \frac{n_1}{\alpha+n}, \dots, \frac{n_{c_n}}{\alpha+n}, \frac{\alpha/S}{\alpha+n} \otimes \mathbf{1}_S), \quad (13c)$$

$$\alpha | \mathbf{Z}_n \sim \pi(\alpha | c_n), \quad (13d)$$

so that $\hat{\beta}(F^*)$ is a WLS estimator, with weights given by the multinomial random variable draw, $\mathbf{n}^* = (n_c^*)_{c=1}^{c_n+S}$, scaled by $\frac{n+\alpha+1}{n+\text{ceil}(\alpha)+1}$, where \otimes is the Kronecker product operator. Similarly, Lancaster (2003) showed that a random OLS functional is also a WLS estimator, in the context of Efron's bootstrap. Also, the current bootstrap sampling scheme (13) implicitly samples from the MDP baseline distribution (F_0) because the bottom S rows of (\mathbb{X}, \mathbb{Y}) already consist of the imaginary observations sampled from F_0 (1c) (see Section 3).

Given (\mathbf{Z}_n, α) , the random variate \mathbf{n}^* (13b) has mean (\mathbb{E}) and covariance matrix (\mathbb{V}):

$$\mathbb{E}(\mathbf{n}^* | \mathbf{Z}_n, \alpha) = \bar{\mathbf{n}}_\alpha^* = (n + \alpha + 1) \left(\frac{n_1}{\alpha+n}, \dots, \frac{n_{c_n}}{\alpha+n}, \frac{\alpha/S}{\alpha+n} \otimes \mathbf{1}_S \right)^\top, \quad (14a)$$

$$\mathbb{V}(\mathbf{n}^* | \mathbf{Z}_n, \alpha) = \text{diag}(\bar{\mathbf{n}}_\alpha^*) - (n + \alpha + 1)^{-1} \bar{\mathbf{n}}_\alpha^* \bar{\mathbf{n}}_\alpha^{*\top}. \quad (14b)$$

Again, in the case of the ridge baseline prior, we can use $(\mathbb{X}, \mathbb{Y}) = \left(\begin{matrix} \mathbf{X}_{c_n}^* \\ \text{diag}(0, v_{x2}, \dots, v_K)^{1/2}, \mathbf{y}_{c_n}^* \\ \mathbf{0}_K \end{matrix} \right)$ and use $\frac{\alpha}{\alpha+n} \otimes \mathbf{1}_K^\top$ in place of $\frac{\alpha/S}{\alpha+n} \otimes \mathbf{1}_S^\top$, in (13a) and (14a).

Let A be a fine grid of α defined over the support of the prior, $\pi(\alpha)$. Then, marginalizing over the posterior $\pi(\alpha | c_n)$ (in (7)), and by the total law of probability for expectations and covariances, the marginal expectation and covariance matrix can be approximated and rapidly computed by:

$$\mathbb{E}(\mathbf{n}^* | \mathbf{Z}_n) = \bar{\mathbf{n}}^* \approx \sum_{\alpha \in A} \mathbb{E}(\mathbf{n}^* | \mathbf{Z}_n, \alpha) \pi(\alpha | c_n), \quad (15a)$$

$$\mathbb{V}(\mathbf{n}^* | \mathbf{Z}_n) \approx \sum_{\alpha \in A} \mathbb{V}(\mathbf{n}^* | \mathbf{Z}_n, \alpha) \pi(\alpha | c_n) + \sum_{\alpha \in A} \mathbb{E}(\mathbf{n}^* | \mathbf{Z}_n, \alpha) \{\mathbb{E}(\mathbf{n}^* | \mathbf{Z}_n, \alpha)\}^\top \pi(\alpha | c_n) \quad (15b)$$

$$- \mathbb{E}(\mathbf{n}^* | \mathbf{Z}_n) \{\mathbb{E}(\mathbf{n}^* | \mathbf{Z}_n)\}^\top. \quad (15c)$$

We have found that the quantities above are rather robust to choice of fine grid A . We assume that the values of the grid A are equally-spaced by .005, with minimum .005 and maximum ξ .

We now consider a deterministic approach to evaluating the distribution of a functional (e.g., $\beta^*(F^*)$) of the MDP posterior, as in previous research on DP functionals (Regazzini, et al. 2002). Specifically, here we employ the multivariate delta method to approximate the (MDP bootstrap) posterior distribution of $\beta(F^*)$ via a Taylor series approximation $\beta^*(\mathbf{n}^*) \approx \beta(\bar{\mathbf{n}}^*)$ of $\beta(\mathbf{n}^*)$ around the mean, $\bar{\mathbf{n}}^*$. With $\frac{\partial \beta(\mathbf{n})}{\partial \mathbf{n}}$ a $K \times (c_n + S)$ matrix of first derivatives (again, $S = K$ for the ridge baseline prior), this Taylor series approximation is given by:

$$\beta^*(\mathbf{n}^*) = \beta(\bar{\mathbf{n}}^*) + \left[\frac{\partial \beta(\mathbf{n}^*)}{\partial \mathbf{n}^*} \right]_{\mathbf{n}=\bar{\mathbf{n}}^*} (\mathbf{n}^* - \bar{\mathbf{n}}^*) \quad (16a)$$

$$= \beta(\bar{\mathbf{n}}^*) + [(\mathbb{Y} - \mathbb{X}\beta(\bar{\mathbf{n}}^*))^\top \otimes (\mathbb{X}^\top \text{diag}(\bar{\mathbf{n}}^*) \mathbb{X})^{-1} \mathbb{X}^\top] \left[\frac{\partial \text{vec}\{\text{diag}(\mathbf{n})\}}{\partial (\mathbf{n})^\top} \right]_{\mathbf{n}=\bar{\mathbf{n}}^*} (\mathbf{n} - \bar{\mathbf{n}}^*) \quad (16b)$$

$$= \beta(\bar{\mathbf{n}}^*) + [\mathbf{u}(\bar{\mathbf{n}}^*)^\top \otimes \{\mathbb{X}(\mathbb{X}^\top \text{diag}(\bar{\mathbf{n}}^*) \mathbb{X})^{-1}\}^\top] \begin{bmatrix} \mathbf{e}_1 \mathbf{e}_1^\top \\ \vdots \\ \mathbf{e}_{c_n+S} \mathbf{e}_{c_n+S}^\top \end{bmatrix} (\mathbf{n} - \bar{\mathbf{n}}^*) \quad (16c)$$

$$= \beta(\bar{\mathbf{n}}^*) + \mathbf{R}(\bar{\mathbf{n}}^*)^\top (\mathbf{n}^* - \bar{\mathbf{n}}^*), \quad (16d)$$

which is similar but not identical to Poirier's (2011, p. 461) approximation, where:

$$\mathbf{u}(\bar{\mathbf{n}}^*) = \mathbb{Y} - \mathbb{X}\beta(\bar{\mathbf{n}}^*); \quad \mathbf{e}_c = (\mathbf{1}(c=1), \dots, \mathbf{1}(c=c_n+S))^\top; \quad (17a)$$

$$\mathbf{R}(\bar{\mathbf{n}}^*) = (\mathbf{e}_1 \mathbf{e}_1^\top, \dots, \mathbf{e}_{c_n+S} \mathbf{e}_{c_n+S}^\top) (\mathbf{u}(\bar{\mathbf{n}}^*) \otimes \mathbb{X}(\mathbb{X}^\top \text{diag}(\bar{\mathbf{n}}^*) \mathbb{X})^{-1}) \quad (17b)$$

$$= \text{diag}\{\mathbf{u}(\bar{\mathbf{n}}^*)\} \mathbb{X}(\mathbb{X}^\top \text{diag}(\bar{\mathbf{n}}^*) \mathbb{X})^{-1}, \text{ is } (c_n + S) \times K. \quad (17c)$$

Then the posterior distribution of \mathbf{n}^* implies that the approximation (16) has exact posterior mean given by the WLS estimator:

$$\mathbb{E}(\beta^*(\mathbf{n}^*) | \mathbf{Z}_n) = \beta(\bar{\mathbf{n}}^*) = (\mathbb{X}^\top \text{diag}(\bar{\mathbf{n}}^*) \mathbb{X})^{-1} \mathbb{X}^\top \text{diag}(\bar{\mathbf{n}}^*) \mathbb{Y}, \quad (18)$$

and exact covariance matrix given by the heteroscedastic-consistent sandwich estimator for WLS (Greene, 2012, p. 319):

$$\mathbb{V}(\beta^*(\mathbf{n}^*) | \mathbf{Z}_n) = \left[\frac{\partial \beta(\mathbf{n}^*)}{\partial \mathbf{n}^*} \right]_{\mathbf{n}=\bar{\mathbf{n}}^*} \mathbb{V}(\mathbf{n}^* | \mathbf{Z}_n) \left[\frac{\partial \beta(\mathbf{n}^*)}{\partial \mathbf{n}^*} \right]_{\mathbf{n}=\bar{\mathbf{n}}^*}^\top = \mathbf{R}(\bar{\mathbf{n}}^*)^\top \mathbb{V}(\mathbf{n}^* | \mathbf{Z}_n) \mathbf{R}(\bar{\mathbf{n}}^*) \quad (19a)$$

$$= (\mathbb{X}^\top \text{diag}(\bar{\mathbf{n}}^*) \mathbb{X})^{-1} [\mathbb{X}^\top \text{diag}(\bar{\mathbf{n}}^* \circ \{\mathbf{u}(\bar{\mathbf{n}}^*)\}^2) \mathbb{X}] (\mathbb{X}^\top \text{diag}(\bar{\mathbf{n}}^*) \mathbb{X})^{-1}, \quad (19b)$$

where \circ denotes the Hadamard product operator. Then the posterior variances from this matrix,

$$\text{diag}\{\mathbb{V}(\beta^*(\mathbf{n}) | \mathbf{Z}_n)\} = \{\mathbb{V}(\beta_1^*(\mathbf{n}^*) | \mathbf{Z}_n), \dots, \mathbb{V}(\beta_K^*(\mathbf{n}^*) | \mathbf{Z}_n)\},$$

provide the asymptotic-consistent 95% posterior credible interval, $\beta_k^*(\mathbf{n}^*) \pm 1.96\{\mathbb{V}(\beta_k^*(\mathbf{n}^*) | \mathbf{Z}_n)\}^{1/2}$, respectively for $k = 1, \dots, K$.

For fixed α , we can write the expectation (18) as $\beta(\bar{\mathbf{n}}_\alpha^*)$, and write the posterior covariance matrix (19) as $\mathbb{V}(\beta^*(\mathbf{n}_\alpha^*) | \mathbf{Z}_n)$.

Suppose that the MDP model (1) assumes a non-informative DP prior, defined by the specification $\alpha \rightarrow 0$. Also suppose that $c_n = n$, so that $\bar{\mathbf{n}} = \bar{\mathbf{n}}_0 = (\mathbf{1}_n^\top, \mathbf{0}_S^\top)^\top$ with $\mathbf{0}_S$ a $S \times 1$ vector of zeros ($S = K$ for the ridge baseline prior). Then the conditional posterior distribution (5) is Dirichlet (Di), $\theta_{c_n} = (\theta_1, \dots, \theta_{c_n})^\top | \mathbf{Z}_n \sim \text{Di}_{c_n}(\mathbf{n}_{c_n})$, with support points the c_n observed cluster values $\{\mathbf{z}_c^*\}_{c=1}^{c_n \leq n}$, where $\theta_c = \Pr\{\mathbf{z} = \mathbf{z}_c^*\}$ for $c = 1, \dots, c_n$, which coincides with the posterior distribution of sampling probabilities under the non-informative Bayesian Bootstrap (Rubin, 1981). Then the posterior predictive probability distribution (6a)-(6c) reduces to $\Pr(\mathbf{z}_{n+1} \in B | \mathbf{Z}_n) = \hat{F}_n(B) = \sum_{c=1}^{c_n} \frac{n_c}{n} \delta_{\mathbf{z}_c^*}(B)$, which is the distribution function employed by the classical (pairs) bootstrap (Efron, 1979); the posterior mean (18) is nearly equal to the OLS estimator, with $\beta(\bar{\mathbf{n}}_0) \approx (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}$ and $\bar{\mathbf{n}}_0 = \lim_{\alpha \rightarrow 0} \bar{\mathbf{n}}_\alpha$; and the posterior covariance matrix (19) is nearly equal to the heteroscedasticity consistent (sandwich) covariance matrix estimator of White (1980) using total sample size weight of $(n+1)$, with:

$$\mathbb{V}(\beta^*(\mathbf{n}^*) | \mathbf{Z}_n) = (\mathbf{X}^\top \mathbf{X})^{-1} [\mathbf{X}^\top \text{diag}(\{\mathbf{u}(\bar{\mathbf{n}}_0^*)\}^2) \mathbf{X}] (\mathbf{X}^\top \mathbf{X})^{-1} \approx \text{HC0}. \quad (20)$$

5 Sensitivity and VoE Analysis Methods

We propose and describe sensitivity analysis methods that can be applied in settings where the assumption of exogeneity may be empirically violated. Suppose that the following linear regression equation holds true for a given population:

$$y_i = \beta_0 + \mathbf{x}_i^\top \beta_{\mathbf{x}} + \beta_T t_i + \gamma u_i + \varepsilon_i = \mathbb{E}(Y | \mathbf{x}_i, t_i, u_i) + \varepsilon_i, \quad i = 1, \dots, n, \quad (21)$$

where $\beta = (\beta_0, \beta_{\mathbf{x}}^\top, \beta_T)^\top$; β_T is the true causal effect of a binary (0,1) treatment variable T on Y ; γ is a possibly non-zero effect of U on Y ; T may be correlated with U ; and the (\mathbf{x}_i, t_i, u_i) are realizations of the random variables $\mathbf{X}_i^\top = (\mathbf{x}_i^\top, T_i)$, and U (resp.).

Suppose for this population that the statistician misspecifies (21) by the regression equation:

$$y_i = \beta_0 + \mathbf{x}_i^\top \tilde{\beta}_{\mathbf{x}} + \tilde{\beta}_T t_i + \varepsilon_{iU} = \mathbb{E}(Y | \mathbf{x}_i, t_i) + \varepsilon_{iU}, \quad i = 1, \dots, n, \quad (22)$$

where U is a missing variable, $\varepsilon_{iU} = \gamma u_i + \varepsilon_i$ for $i = 1, \dots, n$. Then T violates the exogeneity assumption (i.e., is endogenous) if it is correlated with the error term ε_U , making U a source of hidden bias (Rosenbaum, 2002); the OLS (WLS) estimator of $\tilde{\beta}_T$ is inconsistent for β_T (Greene, 2012, p. 259); and the coefficients $\tilde{\beta} = (\tilde{\beta}_0, \tilde{\beta}_{\mathbf{x}}^\top, \tilde{\beta}_T)^\top$ attain the status as pseudo parameters, having covariance matrix that can still be consistently estimated by the sandwich estimator.

Assuming no interactions between (T, U, \mathbf{X}) , the relationship between β_T and $\tilde{\beta}_T$ is given by:

$$\beta_T = \tilde{\beta}_T - \gamma \int \int \{u dF_U(u | T = 1, \mathbf{x}) - u dF_U(u | T = 0, \mathbf{x})\} dF_{\mathbf{X}}(\mathbf{x}), \quad (23)$$

where μ_1 and μ_0 is the mean under distribution (c.d.f.) $F_U(u | T = 1, \mathbf{x})$ and $F_U(u | T = 0, \mathbf{x})$ (resp.); and further, if $\mathbb{E}(U | \mathbf{x}, t) = \mu_{t, \mathbf{x}} = \mu_t + q(\mathbf{x})$ for some function q , then $\beta_T = \tilde{\beta}_T - \gamma(\mu_1 - \mu_0)$

(VanderWeele & Arah, 2011, Appendix). Also, the missing variable, U , may be assumed to have a binomial distribution $F_U(u|t, \underline{\mathbf{x}})$ with success probability $\mu_{t, \underline{\mathbf{x}}}$, with no loss of generality (Wang & Kreiger, 2006).

Along these lines, a new Vibration of Effects (VoE) analysis method, described here, can also be employed for sensitivity analysis. Specifically, this method provides a way to evaluate how the heteroscedastic-consistent effect size of the treatment variable, given by $\widetilde{ES}_{T\alpha} = \widetilde{\beta}_T^* / \mathbb{V}_{T\alpha}^{1/2} = \widetilde{\beta}_T^*(\mathbf{n}_\alpha^*) / \{\mathbb{V}(\beta_T^*(\mathbf{n}_\alpha^*) | \mathbf{Z}_n, \alpha)\}^{1/2}$, varies as a function of the other covariates that are included in the regression model, and α . To explain this method, assume for the MDP model the ridge baseline prior (2) with prior covariances $\mathbb{V} = \text{diag}(0, \mathbf{1}_{K-1}^\top, 0)$, with little loss of generality. Then, given α , and after rescaling each of the columns of $\left[\begin{pmatrix} \mathbf{X}_0 \\ \sqrt{\alpha} \mathbf{I}_K \end{pmatrix}, \begin{pmatrix} \mathbf{y} \\ \mathbf{0}_K \end{pmatrix} \right]$ to have mean zero and variance 1, yielding $(\widetilde{\mathbf{X}}_\alpha, \widetilde{\mathbf{Y}})$, where \mathbf{X}_0 is \mathbf{X} after removing the first column of 1s, the LARS algorithm is run on $(\widetilde{\mathbf{X}}_\alpha, \widetilde{\mathbf{Y}})$ in order to obtain a sequence of estimates $\widehat{\beta}_{0, \text{lar}}^{(\alpha)}, \widehat{\beta}_{1, \text{lar}}^{(\alpha)}, \dots, \widehat{\beta}_{k, \text{lar}}^{(\alpha)}, \dots, \widehat{\beta}_{K-1, \text{lar}}^{(\alpha)}$, where for $k = 0, \dots, K-2$, $\widehat{\beta}_{k, \text{lar}}^{(\alpha)}$ is the LARS estimate (10) of the coefficients that contains the best k out of the total $K-1$ covariates in the regression equation, given α . Then for each subset $\mathcal{S}_{Tl}^{(\alpha)}$ (for $l = 1, 2, \dots$) of the K covariate subsets that includes the treatment variable T , and now using $(\mathbb{X}, \mathbb{Y})_\alpha = \left[\begin{pmatrix} \mathbf{X}_{c_n}^* \\ \mathbf{0}_K, \sqrt{\alpha} \mathbf{I}_K \end{pmatrix}, \begin{pmatrix} \mathbf{y}_{c_n}^* \\ \mathbf{0}_K \end{pmatrix} \right]$, we compute the WLS estimate $\beta(\mathbf{n}_\alpha^*)$ and the heteroscedastic consistent covariance matrix $\mathbb{V}(\beta^*(\mathbf{n}^*) | \mathbf{Z}_n, \alpha)$, using (18) and (19) (rep.), in order to obtain the effect size estimate $\widetilde{ES}_{T\alpha}(\mathcal{S}_{Tl}^{(\alpha)})$, for $l = 1, 2, \dots$. This procedure involving LARS and subsequent $\widetilde{ES}_{T\alpha}(\mathcal{S}_{Tl}^{(\alpha)})$ estimation, for each covariate subset $\mathcal{S}_{Tl}^{(\alpha)}$, is run for each value of α over a fine grid A of values in the support of the prior $\pi(\alpha)$. The entire procedure yields a large collection of effect size $\widetilde{ES}_{T\alpha}(\mathcal{S}_{Tl}^{(\alpha)})$ statistics over all relevant covariate subsets $\mathcal{S}_{Tl}^{(\alpha)}$ given α , over the grid A of α values, to provide a VoE analysis of the heteroscedastic-consistent effect size, \widetilde{ES}_T . These effect sizes $\widetilde{ES}_{T\alpha}(\mathcal{S}_{Tl}^{(\alpha)})$ can be associated with values of the Generalized Information Criterion, $GIC_2(\alpha, p) = \frac{1}{(n+\alpha)} \{ \|\mathbb{Y} - \mathbb{X}\beta(\mathbf{n}_\alpha^*)\|^2 + 2p \}$ (Fan & Tang, 2013), indicating the quality of the predictive fit of the regression that includes p covariates and penalty $2p$. Effect sizes associated with smaller values of GIC_2 may then receive higher priority for statistical inference.

Moreover, using (23), and a binary missing confounding variable U , we may additionally compute and observe the effect size estimator $ES_{T\alpha}(\gamma, \boldsymbol{\lambda}) = \beta_T / \mathbb{V}_{T\alpha}^{1/2}$, over independent standard normal random samples of $(\gamma, \boldsymbol{\lambda})$, where $F_U(u|T=1, \underline{\mathbf{x}}; \boldsymbol{\lambda})$ and $F_U(u|T=0, \underline{\mathbf{x}}; \boldsymbol{\lambda})$ are specified by a binary logistic regression of U on $(T, \underline{\mathbf{x}})$ with coefficients $\boldsymbol{\lambda}$. (If the observations of $(T, \underline{\mathbf{X}}^\top)$ were zero-mean centered before WLS estimation, then these two binary regressions would be performed conditionally on the maximum and minimum values of the zero-centered T , resp.). Section 7 illustrates this entire VoE method through the analysis of two real data sets.

6 Simulation Study

A simulation study was performed to compare the coverage rates of the 95% heteroscedastic-consistent posterior intervals of the coefficient of a covariate, obtained from three models (resp.). They include the MDP model specified under a uniform $\text{un}(\alpha | 0, 3)$ prior, the MDP model under the ξ -truncated Cauchy-type prior for α , with $\xi = 3$ (Section 2); and the linear model estimated under OLS using White's original sandwich covariance estimator (HC0) (9). Also, for each MDP model, we assumed the unit ridge baseline prior. Then, α is the coefficient shrinkage penalty parameter of ordinary ridge regression (Section 3), and the standard HC0 model assumes $\alpha = 0$ (Section 4). Before fitting each model to each simulated data set, the covariate data were zero-mean centered.

X dist.	n	Heteroscedasticity Level											
		$a_h = 0$ (.0)			$a_h = 1$ (.05)			$a_h = 2$ (.1)			$a_h = 2.5$ (.15)		
		c	u	h	c	u	h	c	u	h	c	u	h
U(0,1)	10	.59	.58	.83	.72	.71	.82	.87	.87	.79	.93	.92	.78
Ex(1)	10	.78	.78	.79	.74	.74	.73	.70	.69	.68	.66	.66	.64
N(0,25)	10	.81	.81	.82	.65	.65	.67	.64	.64	.67	.66	.66	.70
AR(1)	10	.82	.82	.81	.81	.80	.80	.78	.79	.78	.77	.77	.77
U(0,1)	20	.74	.74	.90	.84	.84	.89	.92	.92	.87	.95	.95	.86
Ex(1)	20	.86	.86	.86	.81	.81	.80	.75	.75	.75	.71	.71	.70
N(0,25)	20	.88	.88	.88	.84	.84	.85	.88	.88	.89	.90	.90	.91
AR(1)	20	.88	.88	.89	.87	.87	.87	.86	.86	.86	.85	.85	.86
U(0,1)	50	.88	.88	.93	.92	.92	.93	.94	.94	.92	.94	.94	.92
Ex(1)	50	.90	.90	.91	.86	.86	.86	.82	.82	.82	.82	.82	.82
N(0,25)	50	.92	.92	.92	.93	.93	.93	.96	.96	.97	.98	.98	.98
AR(1)	50	.93	.93	.93	.92	.92	.92	.91	.91	.92	.91	.91	.92
U(0,1)	100	.92	.92	.94	.94	.94	.94	.94	.94	.94	.94	.94	.94
Ex(1)	100	.92	.92	.92	.89	.89	.90	.89	.89	.89	.90	.90	.90
N(0,25)	100	.94	.94	.94	.96	.96	.96	.98	.98	.98	.99	.99	.99
AR(1)	100	.94	.94	.94	.93	.93	.94	.93	.93	.93	.93	.93	.93
U(0,1)	500	.94	.94	.95	.95	.95	.95	.95	.95	.95	.94	.94	.95
Ex(1)	500	.95	.95	.95	.93	.93	.93	.95	.95	.95	.96	.96	.96
N(0,25)	500	.95	.95	.95	.98	.98	.98	1.0	1.0	1.0	1.0	1.0	1.0
AR(1)	500	.95	.95	.95	.95	.95	.95	.95	.95	.95	.95	.95	.95

Table 1: For the 80 simulation conditions, coverage rates of 95 percent posterior (confidence) interval for: c = MDP with Cauchy type prior; u = MDP with uniform prior; and h = HC0. Heteroscedasticity levels 1, 2, 3, and 4 refer to $a = 0, 1, 2, 2.5$ for the U(0,1) covariate distribution (resp.), and refer to $a = 0, .05, .1, .15$ otherwise (resp.).

The simulation study employed a $4 \times 4 \times 5$ design that reflects a wide range of conditions that has been considered in past related research. Each of the 80 total cells of the design used 10K simulated data sets, for a total of 800K. Each data set was simulated by taking n samples from the normal linear model, $y_i | x_i, \beta, \sigma_i^2 \sim N(\beta_0 + \beta_1 x_i, \sigma_i^2)$, with $\beta_0 = \beta_1 = 1$ and $\sigma_i^2 = \exp(a_h x_i + a_h x_i^2)$, for $i = 1, \dots, n$ (as in Cribari-Neto et al., 2000), according to one of four covariate x_i sampling distributions; one of four levels a_h of heteroscedasticity; and one of five sample sizes, $n = 10, 20, 50, 100$, and 500. The four covariate distributions are given by the standard uniform distribution U(0, 1) (Cribari-Neto et al., 2000), the normal N(0, 25) distribution (Cameron & Trivedi, 2005, p. 84), the exponential Ex(1) distribution (Hoff & Wakefield, 2013), and the order-1 auto-regressive AR(1) model with Student errors (Hansen, 2007), i.e., $x_i = 1 + .5x_{i-1} + \varepsilon_i$, $\varepsilon_i \sim \text{St}(0, 1, n - 1)$, for $i = 1, \dots, n$. The four heteroscedasticity levels are given by $a_h = 0, 1, 2, 2.5$ for the U(0, 1) covariate distribution, and otherwise given by $a_h = 0, .05, .1, .15$ (Cribari-Neto et al., 2000), where in each case $a_h = 0$ refers to a condition of homoscedasticity.

Table 1 presents the coverage rates of the 95% heteroscedastic-consistent posterior intervals for the true data-generating slope coefficient (β_1), for each of the 80 cells and the three models. Each rate shown is the proportion of times a model’s estimated 95% interval contained the true data-generating slope value ($\beta_1 = 1$) over the 10K simulated data sets. Table 2 summarizes the coverage rates of Table 1 by averages and standard deviations, stratified by covariate distribution, heteroscedasticity level, and sample size condition. Both tables show that the coverage rates are

	Heteroscedasticity Level											
	$a_h = 0$ (.0)			$a_h = 1$ (.05)			$a_h = 2$ (.1)			$a_h = 2.5$ (.15)		
	c	u	h	c	u	h	c	u	h	c	u	h
U(0,1)	.81 (.13)	.81 (.14)	.91 (.04)	.87 (.09)	.87 (.09)	.91 (.05)	.92 (.03)	.92 (.03)	.89 (.06)	.94 (.01)	.94 (.01)	.89 (.06)
Ex(1)	.88 (.06)	.88 (.06)	.89 (.06)	.85 (.07)	.85 (.07)	.84 (.07)	.82 (.09)	.82 (.09)	.82 (.09)	.81 (.11)	.81 (.11)	.80 (.12)
N(0,25)	.90 (.05)	.90 (.05)	.90 (.05)	.87 (.12)	.87 (.12)	.88 (.11)	.89 (.13)	.89 (.13)	.90 (.12)	.91 (.13)	.91 (.13)	.92 (.11)
AR(1)	.90 (.05)	.90 (.05)	.90 (.05)	.90 (.05)	.90 (.05)	.90 (.05)	.89 (.06)	.89 (.06)	.89 (.06)	.88 (.07)	.88 (.07)	.88 (.07)
$n = 10$.75 (.09)	.75 (.10)	.81 (.01)	.73 (.06)	.72 (.06)	.76 (.06)	.75 (.09)	.75 (.09)	.73 (.06)	.75 (.11)	.75 (.11)	.72 (.06)
$n = 20$.84 (.06)	.84 (.06)	.88 (.01)	.84 (.02)	.84 (.02)	.85 (.03)	.85 (.06)	.85 (.06)	.84 (.06)	.85 (.09)	.85 (.09)	.83 (.08)
$n = 50$.91 (.02)	.91 (.02)	.92 (.01)	.91 (.03)	.91 (.03)	.91 (.03)	.91 (.05)	.91 (.05)	.91 (.05)	.91 (.06)	.91 (.06)	.91 (.06)
$n = 100$.93 (.01)	.93 (.01)	.94 (.01)	.93 (.02)	.93 (.02)	.93 (.02)	.94 (.03)	.94 (.03)	.94 (.03)	.94 (.03)	.94 (.03)	.94 (.03)
$n = 500$.95 (.00)	.95 (.00)	.95 (.00)	.95 (.02)	.95 (.02)	.95 (.02)	.96 (.02)	.96 (.02)	.96 (.02)	.96 (.02)	.96 (.02)	.96 (.02)
Total	.87 (.09)	.87 (.09)	.90 (.05)	.87 (.09)	.87 (.09)	.88 (.08)	.88 (.10)	.88 (.10)	.88 (.09)	.88 (.10)	.88 (.10)	.87 (.10)

Table 2: Means (standard deviations) of the simulation results of Table 1.

generally similar across the three models, especially for the larger sample size conditions, where the coverage rates for all three models approach .95. As Table 2 shows for the uniform $U(0, 1)$ covariate distribution, HC0 tended to be closer to .95 for the lower two heteroscedasticity levels, whereas the converse was true for the higher two heteroscedasticity levels. The same was true for the $n = 10$ and $n = 20$ sample size conditions.

However, recall that the simulation study focused on the generation of positive-definite covariate design matrices (\mathbf{X}). A design matrix that has multicollinearity or is singular can yield an infinite value of HC0, whereas for a MDP model with ridge baseline prior will still yield a defined posterior covariance matrix. This is known to be an advantage of ridge regression over OLS regression.

7 Real Data Applications

We now illustrate the application of the MDP model on two real data sets, assuming unit ridge baseline prior, and a uniform $\text{un}(\alpha | 0, 3)$ prior for α .

7.1 LMT Data

Here we analyze real data set of observations from $n = 347$ undergraduate teacher education students (89.9% female) who each attended one of four Chicago universities between the Fall 2007 semester and Fall 2013 spring semesters, inclusive, excluding summers. The primary aim of the analysis is to infer the effect of the new teacher education curriculum (versus old curriculum) on

a dependent variable (Y) of math teaching ability. Here, ability is defined as the number-correct score obtained on a 25-item test of Learning Math for Teaching (LMT, 2012), after completing a course on algebra teaching. Three covariates were considered, namely, Year and Year², and CTPP = $\mathbf{1}(\text{Year} \geq 2010.9)$, an indicator of the administration of the new (versus old) teaching curriculum. All covariates were rescaled to have mean zero and variance 1 before data analysis.

	$\beta(\bar{\mathbf{n}}^*)$	pSD	ES	95%PI $\beta(\bar{\mathbf{n}}^*)$	OLS $\hat{\beta}$	SE
Intercept	12.90	.18	72.76	(12.55, 13.24)	12.90	.18
Year	-.69	.57	-1.21	(-1.81, .43)	-478.17	426.54
Year ²	-.65	.57	-1.14	(-1.78, .47)	476.75	426.54
CTPP	.60	.28	2.10	(.04, 1.15)	.67	.28

Table 3: The slope coefficient estimates for the real data set, including heteroscedastic-consistent posterior standard deviation (pSD) and robust 95 percent posterior intervals (PI). Also included are the OLS estimates of the coefficients and their respective robust standard errors (SE).

Figure 1

Table 3 presents the results of the data analysis, in terms of the MDP-based WLS estimates ($\beta(\bar{\mathbf{n}}^*)$) and their respective heteroscedastic-consistent (robust) 95% posterior credible intervals. The CTPP causal effect was significant, as this covariate’s 95% heteroscedastic-consistent posterior interval (.04, 1.15) excludes zero. This table also presents the OLS estimates ($\hat{\beta}$) and their respective robust standard errors based on the ordinary sandwich estimator, and show that the OLS estimate of the Year slope coefficient and its standard error are large (in absolute value) due to the multicollinearity of the Year and Year² covariate observations. This is not true for any of the WLS estimates and corresponding posterior standard deviations (pSD). As mentioned, multicollinearity can explode the variance of OLS estimates. In contrast, in ridge regression, provided by the MDP ridge baseline prior, helps control the size of the WLS and variance estimates of the coefficients by shrinking coefficient estimates towards zero.

Figure 1 presents the results of the VoE analysis, relating the CTPP effect size, GIC_2 , α , and subsets of the covariates (Year, Year², CTPP) chosen by the LARS algorithm, only among the subsets that included CTPP. These results are based on a total of 605 regressions (CTPP effect sizes). Over these conditions, the CTPP effect is rather stable. The figure also presents a sensitivity analysis of a hypothetical missing variable U , over 50 independent standard normal random samples of (γ, λ) , and shows some instability of the CTPP effect size with respect to this variable.

7.2 PIRLS Data

A data set was obtained from the 2006 Progress in International Reading Literacy Study (PIRLS), on 565 low-income students from 21 U.S. elementary schools. For data analysis, the dependent variable is student literacy score (zREAD), along with 8 covariates: student male status (1 if MALE, or 0), AGE, class size (SIZE), class percent of English language learners (ELL); teacher years of experience (TEXP4) and education level (EDLEVEL = 5 if bachelor’s; EDLEVEL = 6 if at least master’s degree); school enrollment (ENROL) and safety rating (SAFE = 1 is high; SAFE = 3 is low). Each variable in the data set was rescaled to z-scores having mean 0 and variance 1.

	$\beta(\bar{\mathbf{n}}^*)$	pSD	ES	95%PI $\beta(\bar{\mathbf{n}}^*)$	OLS $\hat{\beta}$	SE
Intercept	-.48	.04	-12.67	(-.56, -.41)	-.48	.04
MALE	-.06	.04	-1.44	(-.13, .02)	-.06	.04
AGE	-.20	.04	-5.22	(-.27, -.12)	-.20	.04
SIZE	-.07	.05	-1.48	(-.16, .02)	-.07	.05
ELL	-.13	.05	-2.81	(-.22, -.04)	-.13	.05
TEXP4	.14	.04	3.23	(.05, .22)	.14	.04
EDLEVEL	.03	.04	.77	(-.05, .11)	.03	.04
ENROL	.30	.04	7.60	(.22, .37)	.30	.04
SAFE	-.16	.04	-3.71	(-.24, -.07)	-.16	.04

Table 4: The slope coefficient estimates for the real data set, including heteroscedastic-consistent posterior standard deviation (pSD) and robust 95 percent posterior intervals (PI). Also included are the OLS estimates of the coefficients and their respective robust standard errors (SE).

Table 4 presents the results of the data analysis, including the MDP-based WLS estimates ($\beta(\bar{\mathbf{n}}^*)$), their respective heteroscedastic-consistent (robust) 95% posterior credible intervals. According to the MDP model, teacher’s years of experience (TEXP4) was found to have a significant effect on reading performance, as its slope coefficient estimate had a robust 95% posterior interval that excluded zero. Figure 2 presents the results of the VoE analysis of the TEXP4 effect size, based on a total of 3,600 regressions (TEXP4 effect sizes). This figure shows that the TEXP4 effect is relatively stable and has an overall tendency to be significant, and the larger TEXP4 effect sizes tend to be associated with better (smaller) *GIC* statistics. The figure also presents results of a sensitivity analysis of a hypothetical missing confounding variable U , over 50 independent standard normal random samples of $(\gamma, \boldsymbol{\lambda})$, and shows instability of the TEXP4 effect after accounting for this variable.

Figure 2

8 Conclusions

This study introduced and illustrated regression methodology that is useful for performing inferences of the mean dependent response. This methodology was developed by establishing new connections between Dirichlet process functional inference, the bootstrap, heteroscedastic-consistent sandwich covariance estimation, ridge shrinkage regression, WLS, and VoE/sensitivity analysis of causal effects. This study is also the first to provide consistent sandwich covariance estimation for ridge regression. A simulation study showed that this MDP/OLS functional methodology is competitive with the sandwich variance estimator in terms of 95% coverage rates of posterior intervals over a large range of conditions. The former estimator has the advantage for observed design matrices (\mathbf{X}) that are multicollinear or singular. Also, the applicability of the regression methodology was illustrated through the analysis of real data, which involves WLS coefficient estimation procedures that are computationally feasible even for very large data sets. A free software package that implements the MDP functional methodology (menu option: "VoE analysis") is available from the author’s website.

Some extensions of the methods of the paper are worthy for future research. The bootstrap approximation methodology of Section 4 yielded explicit closed form equations for the posterior

mean and covariance matrix of the OLS functional of the regression coefficients. This is because equations for $\mathbb{E}(\mathbf{n}^* | \mathbf{Z}_n, \alpha)$ and $\mathbb{V}(\mathbf{n}^* | \mathbf{Z}_n, \alpha)$ in (14) are available in closed form thanks to the conjugacy property of the DP prior. This property not only allows for explicit equations for the mean and variance of the process, but also makes it possible to correspond this mean and variance with those (resp.) of the DP's Pólya urn scheme, the latter which provides the basis for the bootstrap methodology.

In principle, the MDP bootstrap can be extended to other (non-conjugate) Bayesian nonparametric of Gibbs-type (see Leisen & Lijoi, 2011; Bassetti, et al. 2014; Zhu & Leisen, 2015; De Blasi et al., 2015). For each of these other prior processes, the variance $\mathbb{V}(\mathbf{n}^* | \mathbf{Z}_n, \alpha)$ cannot be directly evaluated, because they do not provide explicit characterizations of the process variance. Thus, they would require Monte Carlo simulation methods to implement bootstrap approximations for inferences of process functionals of interest, such as the OLS functional. Finally, the sensitivity analysis methods of Section 5 can be extended to allow for interactions between the treatment variable (T, U, \mathbf{X}) , perhaps by specifying U into the MDP baseline distribution (VanderWeele & Arah, 2011).

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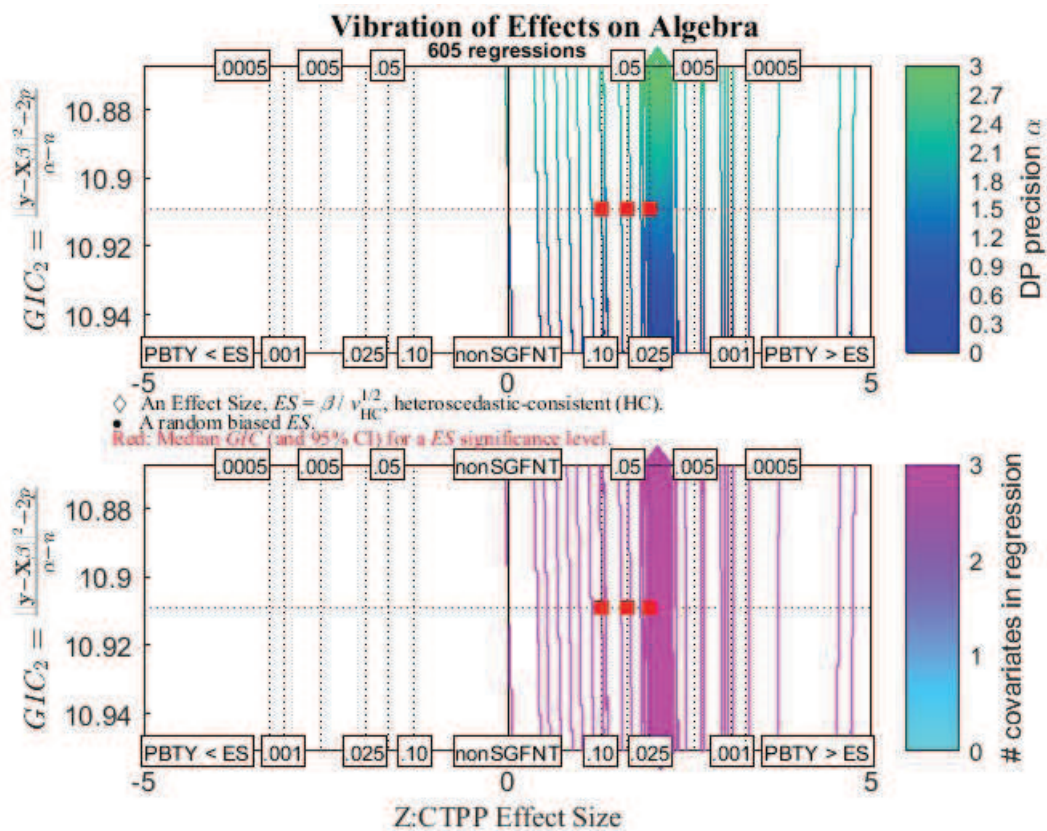


Figure 1. Vibrations of Effects analysis of the LMT data.

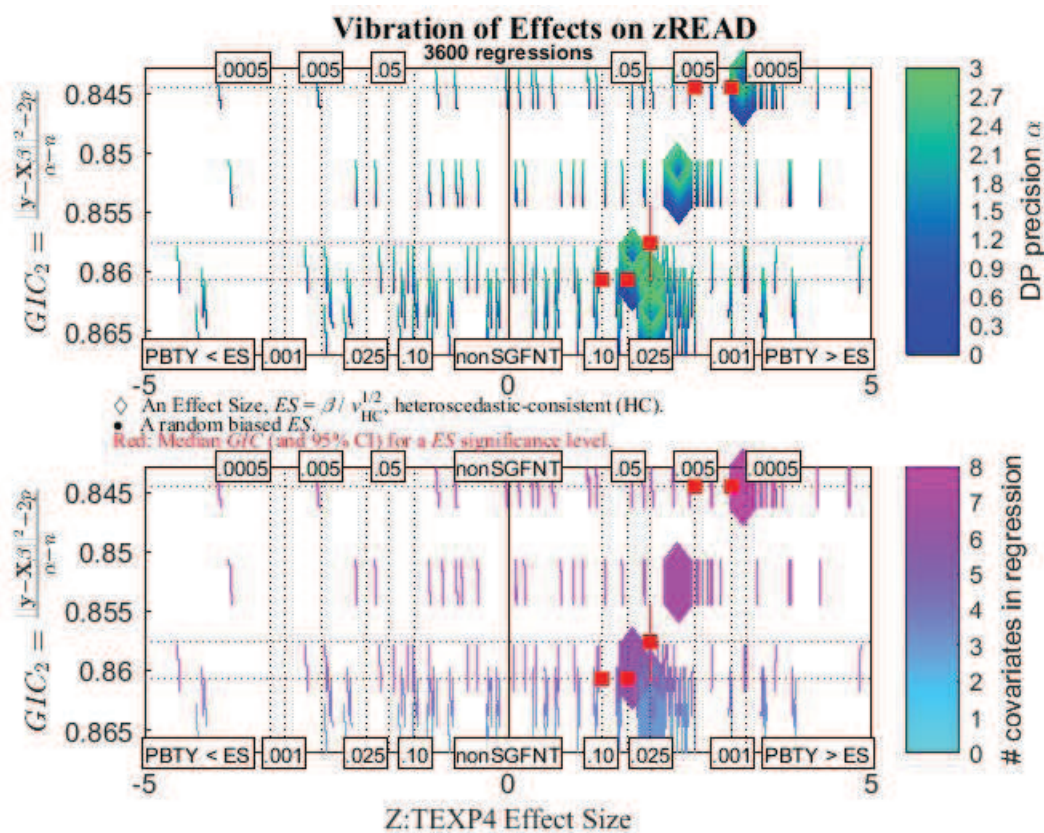


Figure 2. Vibrations of Effects analysis of the PIRLS data.